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Geometric Phase in a Generalized Time-Dependent Karassiov-Klimov Model

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Abstract By using of the invariant theory, we have studied the generalized time-dependent Karassiov-Klimov model, the dynamical and geometric phases are given, respectively. The Aharonov-Anandan phase is also obtained under the cyclical evolution.

Keywords Geometric phase · Time-dependent Karassiov-Klimov model · Invariant theory

1 Introduction

It is known that the concept of geometric phase was first introduced by Pancharatnam [1] in studying the interference of classical light in distinct states of polarization. Berry [2] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [3]. Samuel et al. [4] generalized the pure state geometric phase by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. Mukunda and Simon [5] gave a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out [6]. Recently, the geometric phase of the mixed states has also been studied [7–9].

As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [10] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized by introducing the concept of basic invariants and used to study the geometric

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phases [11–14] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has been developed in some different directions [15–27].

In this paper, by using of the invariant theory, we shall study the dynamical and geometric phases of the generalized time-dependent Karassiov-Klimov model.

2 Model

The Hamiltonian of the generalized time-dependent Karassiov-Klimov model can be written by [28]

$$\hat{H} = \omega_1(t)\hat{a}_1^\dagger\hat{a}_1 + \omega_2(t)\hat{a}_2^\dagger\hat{a}_2 + g(t)\hat{a}_1^{\dagger s}\hat{a}_2^r + g^*(t)\hat{a}_2^{\dagger s}\hat{a}_1^r, \tag{1}$$

where $0 \le r \le s$, $\omega_1(t)$ and $\omega_2(t)$ are the angular frequencies of two independent harmonic oscillators, respectively. $\hat{a}_i^\dagger(\hat{a}_i)$ $(i = 1, 2)$ are the creation (annihilation) operators and satisfying $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. In this letter, we only consider the case of $s = r = 2$. Introducing the following operators

$$J_3 = \frac{1}{4}(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2), \qquad J_+ = \hat{a}_1^{\dagger 2}\hat{a}_2^2, \qquad J_- = \hat{a}_2^{\dagger 2}\hat{a}_1^2, \tag{2}$$

then (1) becomes

$$\hat{H} = [\omega_1(t) + \omega_2(t)]\hat{R} + 2[\omega_1(t) - \omega_2(t)]\hat{J}_3 + g(t)\hat{J}_+ + g^*(t)\hat{J}_-, \tag{3}$$

where $\hat{R} = \frac{1}{2}(a_1^\dagger\hat{a}_1 + a_2^\dagger\hat{a}_2) = \frac{1}{2}(\hat{N}_1 + \hat{N}_2)$. $\hat{J}_3$ and $\hat{J}_\pm$ are the generators of the Higgs algebra, and they satisfy the commutation relations

$$[\hat{J}_3, \hat{J}_\pm] = \pm\hat{J}_\pm, \qquad [\hat{J}_+, \hat{J}_-] = c_1\hat{J}_3 + c_3\hat{J}_3^3, \tag{4}$$

where $c_1 = 8(2R^2 + 2R - 1)$, and $c_3 = -64$.

3 Dynamical and Geometric Phases

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [10]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{I}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial\hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \tag{5}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{6}$$

where $\frac{\partial\lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{7}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (7) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\delta_n(t)]$ for the non-degenerate state, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{8}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (7). Then the general solution of the Schrödinger equation (7) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{9}$$

where

$$\delta_n(t) = \int_0^t dt' \langle\lambda_n, t'|i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t'\rangle, \tag{10}$$

and $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$.

Because $[\hat{R}, \hat{J}_3] = [\hat{R}, \hat{J}_\pm] = 0$, so $[\hat{R}, \hat{H}] = 0$, which means that $\hat{R} = \frac{1}{2}(\hat{N}_1 + \hat{N}_2)$ is a special invariant of this system and satisfies $\hat{R}|m\rangle_{\hat{a}_1}|n\rangle_{\hat{a}_2} = \lambda_{mn}|m\rangle_{\hat{a}_1}|n\rangle_{\hat{a}_2}$, where $\hat{a}_1^\dagger\hat{a}_1|m\rangle_{\hat{a}_1} = m|m\rangle_{\hat{a}_1}$, $\hat{a}_2^\dagger\hat{a}_2|n\rangle_{\hat{a}_2} = n|n\rangle_{\hat{a}_2}$, and $\lambda_{mn} = \frac{1}{2}(m+n)$.

In the following, we can restrict the space being in the sub-space of the eigenstate of the invariant $\hat{R}$, and $\hat{R}$ appeared in (3) can be replaced by its eigenvalue λ_{mn}. So (3) becomes

$$\hat{H} = [\omega_1(t) + \omega_2(t)]\lambda_{mn} + 2[\omega_1(t) - \omega_2(t)]\hat{J}_3 + g(t)\hat{J}_+ + g^*(t)\hat{J}_-. \tag{11}$$

We can introduce the L-R invariant as follows

$$\hat{I} = \cos\theta\hat{J}_3 - e^{-i\varphi}\sin\theta\hat{J}_+ - e^{i\varphi}\sin\theta\hat{J}_-, \tag{12}$$

here $\theta = \theta(t)$ and $\varphi = \varphi(t)$ are determined by (5), and satisfy the auxiliary relations

$$i\dot{\theta} + (\tilde{c_1} - \tilde{c_3}\sin\theta)(g^*e^{-i\varphi} - ge^{i\varphi}) = 0, \tag{13}$$

$$\cos\theta(g - i\dot{\theta}e^{-i\varphi}) + e^{-i\varphi}\sin\theta[2\omega_1 - 2\omega_2 - \dot{\varphi}e^{-i\varphi}] = 0, \tag{14}$$

where dot denotes the time derivative, and $\tilde{c_1} = 8(2\lambda_{mn}^2 + 2\lambda_{mn} - 1)$, $\tilde{c_3} = -(m-n)^3$. Here we have used the large quantum numbers approximation, i.e., $J_3^3 \sim (m-n)^3/64$.

We now construct the unitary transformation

$$\hat{V}(t) = \exp[\sigma\hat{J}_+ - \sigma^*\hat{J}_-], \tag{15}$$

where $\sigma = \frac{\theta}{2}e^{-i\varphi}$ and $\sigma^* = \frac{\theta}{2}e^{i\varphi}$. It is easy to find that when satisfying the following relations

$$[1 - \cos(\sqrt{2\tilde{c_1}}|\sigma|)]\left[\frac{\sigma}{2\sigma^*}e^{i\varphi}\sin\theta + \frac{1}{2}e^{-i\varphi}\sin\theta\right] - e^{-i\varphi}\sin\theta$$
$$+ \frac{\sigma\cos\theta}{\sqrt{2\tilde{c_1}}|\sigma|}[\sqrt{2\tilde{c_1}}|\sigma| - 1 + \sin(\sqrt{2\tilde{c_1}}|\sigma|)] = 0, \tag{16}$$

$$e^{-i\varphi}\sin\theta\frac{\tilde{c_1}\sigma^*}{\sqrt{2\tilde{c_1}}|\sigma|}\sin(\sqrt{2\tilde{c_1}}|\sigma|) + e^{i\varphi}\sin\theta\frac{\tilde{c_1}\sigma}{\sqrt{2\tilde{c_1}}|\sigma|}\sin(\sqrt{2\tilde{c_1}}|\sigma|)$$
$$+ \cos\theta\cos(\sqrt{2\tilde{c_1}}|\sigma|) = 1, \tag{17}$$

$$\frac{\cos\theta}{\tilde{c_1}}[\cos(\sqrt{2\tilde{c_1}}|\sigma|)-1]+\frac{|\sigma|}{\sqrt{2\tilde{c_1}}}\sin\theta\sin(\sqrt{2\tilde{c_1}}|\sigma|)\left[\frac{e^{i\varphi}}{\sigma^*}+\frac{e^{-i\varphi}}{\sigma}\right]=0, \tag{18}$$

then a time-independent invariant $\hat{I}_V$ appears

$$\hat{I}_V=\hat{V}^\dagger(t)\hat{I}\hat{V}(t)=\hat{J}_3. \tag{19}$$

In terms of the unitary transformation $\hat{V}(t)$ and the Baker-Campbell-Hausdoff formula [29]

$$\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}=\frac{\partial\hat{\wp}}{\partial t}+\frac{1}{2!}\left[\frac{\partial\hat{\wp}}{\partial t},\hat{\wp}\right]+\frac{1}{3!}\left[\left[\frac{\partial\hat{\wp}}{\partial t},\hat{\wp}\right],\hat{\wp}\right]+\frac{1}{4!}\left[\left[\left[\frac{\partial\hat{\wp}}{\partial t},\hat{\wp}\right],\hat{\wp}\right],\hat{\wp}\right]+\cdots, \tag{20}$$

where $\hat{V}(t)=\exp[\hat{\wp}(t)]$. It is easy to find that when satisfying the following relation

$$\begin{aligned}&\frac{2(\omega_1-\omega_2)\sigma}{\sqrt{2\tilde{c_1}}|\sigma|}[\sqrt{2\tilde{c_1}}|\sigma|-1+\sin(\sqrt{2\tilde{c_1}}|\sigma|)]+\frac{g}{2}[\cos(\sqrt{2\tilde{c_1}}|\sigma|)-1]+g\\&\quad+\frac{g^*\sigma}{2\sigma^*}[\cos(\sqrt{2\tilde{c_1}}|\sigma|)-1]-\frac{i\sigma(\dot{\sigma}^*\sigma-\dot{\sigma}\sigma^*)}{2\sqrt{2\tilde{c_1}}|\sigma|^3}[\sin(\sqrt{2\tilde{c_1}}|\sigma|)-1]+i\dot{\sigma}=0,\end{aligned} \tag{21}$$

one has

$$\begin{aligned}\hat{H}_V(t)&=\hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}\\&=\left\{2(\omega_1-\omega_2)\cos(\sqrt{2\tilde{c_1}}|\sigma|)+\frac{i(\dot{\sigma}\sigma^*-\dot{\sigma}^*\sigma)}{2|\sigma|^2}[1-\cos(\sqrt{2\tilde{c_1}}|\sigma|)]\right.\\&\quad\left.-\frac{\tilde{c_1}\sin(\sqrt{2\tilde{c_1}}|\sigma|)}{\sqrt{2\tilde{c_1}}|\sigma|}(g\sigma^*+g^*\sigma)\right\}\hat{J}_3\\&\quad+2(\omega_1-\omega_2)[\cos(\sqrt{2\tilde{c_1}}|\sigma|)-1]\frac{\tilde{c_3}}{\tilde{c_1}}-\frac{|\sigma|\tilde{c_3}\sin(\sqrt{2\tilde{c_1}}|\sigma|)}{\sqrt{2\tilde{c_1}}}\left[\frac{g}{\sigma}-\frac{g^*}{\sigma^*}\right]\\&\quad+\frac{i\tilde{c_3}(\dot{\sigma}\sigma^*-\dot{\sigma}^*\sigma)}{2\tilde{c_1}|\sigma|^2}[1-\cos(\sqrt{2\tilde{c_1}}|\sigma|)].\end{aligned} \tag{22}$$

It is easy to find that $\hat{H}(t)$ differs from $\hat{I}_V$ only by a time-dependent c-number factor. Thus we can get the general solution of the time-dependent Schrödinger equation (7)

$$|\Psi(t)\rangle_s=\sum_n\sum_m C_{nm}\exp[i\delta_{nm}(t)]\hat{V}(t)|m\rangle_{\hat{a}_1}|n\rangle_{\hat{a}_2}, \tag{23}$$

with the coefficients $C_{nm}=\langle n,m,t=0|\Psi(0)\rangle_s$. The phase $\delta_{nm}(t)=\delta^d_{nm}(t)+\delta^g_{nm}(t)$ includes the dynamical phase

$$\begin{aligned}\delta^d_{nm}(t)=&-\int_{t_0}^t\frac{1}{4}(m-n)\left\{2(\omega_1-\omega_2)\cos(\sqrt{2\tilde{c_1}}|\sigma|)-\frac{\tilde{c_1}\sin(\sqrt{2\tilde{c_1}}|\sigma|)}{\sqrt{2\tilde{c_1}}|\sigma|}(g\sigma^*+g^*\sigma)\right\}\hat{d}t'\\&-\int_{t_0}^t\left\{2(\omega_1-\omega_2)[\cos(\sqrt{2\tilde{c_1}}|\sigma|)-1]\frac{\tilde{c_3}}{\tilde{c_1}}-\frac{|\sigma|\tilde{c_3}\sin(\sqrt{2\tilde{c_1}}|\sigma|)}{\sqrt{2\tilde{c_1}}}\left[\frac{g}{\sigma}-\frac{g^*}{\sigma^*}\right]\right\}dt',\end{aligned} \tag{24}$$

and the geometric phase

$$\delta^g_{nm}(t) = -i\int_{t_0}^{t}\frac{(m-n)(\dot{\sigma}\sigma^* - \dot{\sigma}^*\sigma)}{8|\sigma|^2}[1-\cos(\sqrt{2\tilde{c_1}}|\sigma|)]dt' \\ - i\int_{t_0}^{t}\frac{\tilde{c_3}(\dot{\sigma}\sigma^* - \dot{\sigma}^*\sigma)}{2\tilde{c_1}|\sigma|^2}[1-\cos(\sqrt{2\tilde{c_1}}|\sigma|)]dt'. \tag{25}$$

Particularly, the geometric phase becomes under the cyclical evolution

$$\delta^g_{nm}(t) = \oint\left[\frac{1}{4}(n-m) - \frac{\tilde{c_3}}{\tilde{c_1}}\right]\left[1-\cos\left(\sqrt{\frac{\tilde{c_1}}{2}}\theta\right)\right]d\varphi, \tag{26}$$

which is the known geometric Aharonov-Anandan phase.

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